Four-dimensional compact solvmanifolds with and without complex analytic structures

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Abstract. We classify four-dimensional compact solvmanifolds up to diffeomorphism, while determining which of them have complex analytic structures. In particular, we shall see that a four-dimensional compact solvmanifold S can be written, up to double covering, as $\Gamma\backslash G$ where G is a simply connected solvable Lie group and Γ is a lattice of G, and every complex structure J on S is the *canonical complex structure* induced from a left-invariant complex structure on G. We are thus led to conjecture that complex analytic structures on compact solvmanifolds are all canonical.

1. INTRODUCTION

In this paper we shall mean by a solvmanifold (nilmanifold) a compact homogeneous space of solvable (nilpotent) Lie group. Let M be an n-dimensional solvmanifold. M can be written as $D\backslash G$, where G is a simply connected solvable Lie group and D is a closed subgroup (which may not be a discrete subgroup) of G. In the case where M is a nilmanifold, we can assume that D is a discrete subgroup of G.

In the author's paper [12] we observed that complex tori and hyperelliptic surfaces are the only four-dimensional solvmanifolds which admit Kähler

structures. In this paper we extend this result to the case of complex structures; namely we will show the following:

Main Theorem. *The complex surfaces with diffeomorphism type of solv-manifolds are all of the complex tori, hyperelliptic surfaces, Primary Kodaira surfaces, Secondary Kodaira surfaces and Inoue surfaces.*

In Section 2 we classify four-dimensional orientable solvmanifolds up to diffeomorphism, which include all of the orientable T^2-bundles over T^2. In Section 3 we study in more details certain classes of four-dimensional solv-manifolds which admit complex structures; we show precisely how to construct complex structures on them. In particular we obtain the classifications of hyperelliptic surfaces and secondary Kodaira surfaces as solvmanifolds. In Section 4 we give the proof of the main theorem. To be more precise, the proof consists of two steps: First, we show that complex surfaces stated above in the main theorem are all obtained by constructing complex structures on certain four-dimensional solvmanifolds. This is the part we shall see in Section 3. Next, we show that a complex surface with diffeomorphism type of solvmanifold must be one of the surfaces stated in the main theorem. This is the part we actually show in Section 4. As a consequence of the theorem together with the results of Section 3, we can see that complex structures on a complex surface S with diffeomorphism type of solvmanifold $\Gamma \backslash G$ (up to finite covering) are exactly those which are canonically induced from left-invariant complex structures on G (Proposition 4.2), where G is a simply connected solvable Lie group and Γ is a lattice of G. In other word these complex structures can be defined simply on the Lie algebras of G (see Section 3.8). In Section 5 (Appendix) we give a complete list of all the complex structures on four-dimensional compact homogeneous spaces, referring to their corresponding complex surfaces. A brief description of the list is provided, though the proof is mostly a combination of many known results, along with the application of Kodaira's classification of complex surfaces.

We shall discuss also some other important structural problems on solv-manifolds: while classifying four-dimensional solvmanifolds we can see the following:

1) Every four-dimensional orientable solvmanifold is real parallelizable (see Proposition 2.3), which is noted in [2] without proof.

2) There exist four-dimensional simply connected, unimodular and non-nilpotent solvable Lie groups which have no lattices (see Section 2.5).

3) There exist four-dimensional solvmanifolds which cannot be written as $\Gamma\backslash G$ where G is a simply connected solvable Lie group and Γ is a lattice of G, although they have the solvmanifolds of the above type as double coverings (see Example 3.5).

2. FOUR-DIMENSIONAL SOLVMANIFOLDS – CLASSIFICATION UP TO DIFFEOMORPHISM

We recall some fundamental results on solvmanifolds. It is well-known [16] that an n-dimensional solvmanifold M is a fiber bundle over a torus with fiber a nilmanifold, which we call the *Mostow fibration* of M. In particular we can represent the fundamental group Γ of M as a group extension of $\mathbf{Z}^k$ by a torsion-free nilpotent group N of rank $n - k$, where $1 \leq k \leq n$ and $k = n$ if and only if Γ is ableain:

$$0 \to N \to \Gamma \to \mathbf{Z}^k \to 0.$$

Conversely any such a group Γ (which is called the *Wang group*) can be the fundamental group of some solvmanifold ([1, 25]). It is also known [16] that two solvmanifolds having isomorphic fundamental groups are diffeomorphic.

From now on we denote by S a four-dimensional orientable solvmanifold, Γ the fundamental group of S, and b_1 the first Betti number of S. The classification is divided into three cases:

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(2.1) $2 \leq k \leq 4$ and N is abelian

(2.2) $k = 1$ and N is abelian

(2.3) $k = 1$ and N is non-abelian

It should be noted that these three cases are not exclusive each other. For instance, it can be easily seen that Γ of the case $k = 3$ can be expressed as the case $k = 2$, and also as the case $k = 1$ and N is either abelian or non-abelian. More precisely, we shall see that the class of solvmanifolds of the case (2.1) coincides with that of T^2-bundles over T^2 (see Proposition 2.1), and most of them belong also to the case (2.2) or (2.3).

2.1. The case $2 \leq k \leq 4$ and N is abelian

In the case $k = 4$, Γ is abelian and S is a 4-torus with $b_1 = 4$. In the case $k = 3$, we have $N = \mathbf{Z}^1$, and since S is orientable the action $\phi : \mathbf{Z}^3 \to \mathrm{Aut}(\mathbf{Z})$ is trivial. Hence Γ is a nilpotent group, which can be canonically extended to the nilpotent Lie group G, and $S = \Gamma \backslash G$ is a nilmanifold with $b_1 = 3$. S has a complex structure defining a primary Kodaira surface (see Section 3.4). In the case $k = 2$, we have $N = \mathbf{Z}^2$, and S is a T^2-bundle over T^2.

We have the following general result (including non-orientable cases), which appears to be well-known.

Proposition 2.1. *The diffeomorphism class of four-dimensional solvmanifolds with $2 \leq b_1 \leq 4$ coincides with that of T^2-bundles over T^2.*

Proof. We know that Γ of the case $k = 3$ can be expressed as the case $k = 2$, and the corresponding nilmaifold with $b_1 = 3$ has a structure of T^2-bundle over T^2. We also know [3] that a solvmanifold of general dimension can has a structure of fiber bundle over T^{b_1} with fiber a nilmanifold (which may be different from the Mostow fibration). Hence, the four-dimensional solvmanifolds with $2 \leq b_1 \leq 4$ are all T^2-bundles over T^2. Conversely, for a given T^2-bundle $\bar{S}$ over T^2 there exits a four-dimensional solvmanfold S with the same fundamental group as $\bar{S}$. Since the diffeomorphism type of T^2-bundle over T^2 is determined only by the fundamental group ([19]), S

must be diffeomorphic to $\bar{S}$. □

The classification of T^2-bundles over T^2 is well-known ([19, 22]). According to the above proposition, we can classify them as four-dimensional solvmanifolds. Let us first consider the group extension:

$$0 \to \mathbf{Z}^2 \to \Gamma \xrightarrow{r} \mathbf{Z}^2 \to 0,$$

where the action $\phi : \mathbf{Z}^2 \to \mathrm{Aut}(\mathbf{Z}^2)$ is defined by $\phi(e_1), \phi(e_2), e_1 = (1,0), e_2 = (0,1)$. Since the total space is orientable $\phi(e_1), \phi(e_2) \in \mathrm{SL}(2, \mathbf{Z})$, and since $\phi(e_1)$ and $\phi(e_2)$ commute we can assume that $\phi(e_1) = \pm \mathrm{I}$ (cf. [19]).

1) In the case where $\phi(e_1) = \mathrm{I}$, by taking a section s of r defined on the first facter of $\mathbf{Z}^2$, we can construct a subgroup $\mathbf{Z}^3$ of Γ with the group extension:

$$0 \to \mathbf{Z}^3 \to \Gamma \to \mathbf{Z}^1 \to 0.$$

Hence this case is included in the case (2,2).

2) In the case where $\phi(e_1) = -\mathrm{I}$, we can assume that the characteristic polynomial Φ of $\phi(e_2)$ has the double root 1 or two distinct positive real roots.

a) If Φ has the double root 1, $\phi(e_2)$ is expressed in the following form:

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

As in the case where $\phi(e_1) = \mathrm{I}$, by taking a section s of r, we can construct a subgroup N of Γ, which is a torsion-free nilpotent group with the group extension:

$$0 \to N \to \Gamma \to \mathbf{Z}^1 \to 0.$$

Hence this case is included in the the case (2,3) (this case is precisely the case 2b) of (2,3)).

b) If Φ has two distinct positive real roots a_1, a_2, we can similarly construct a subgroup K of Γ with the group extension:

$$0 \to K \to \Gamma \to \mathbf{Z}^1 \to 0,$$

where $K = \mathbf{Z}^2 \rtimes \mathbf{Z}$ (a non-nilpotent solvable group) which can be extended to the solvable Lie group $G = \mathbf{R}^2 \rtimes \mathbf{R}$ defined by the action $\psi(t)(x, y) = (e^{t \log a_1} x, e^{t \log a_2} y)$. $\bar{S} = K \backslash G$ is a three-dimensional solvmanifold, and the solvmanifold S corresponding to Γ is a fiber bundle over T^1 with fiber $\bar{S}$ where the action $\phi : \mathbf{Z} \to \mathrm{Aut}(\bar{S})$ is canonically induced by $\phi(m) = (-1)^m \mathrm{I}_2 \times \mathrm{I}_1 \in \mathrm{Aut}(G)$.

Remark 2.2. The solvmanifolds of the case 2a) and 2b) above are the examples which cannot be written as $\Gamma \backslash G$, where G is a simply connected solvable Lie group and Γ is a lattice of G. However, it is clear that they have the solvmanifolds of the above type as double coverings.

2.2. The case $k = 1$ and N is abelian

The group extension is split and thus determined only by the action $\phi : \mathbf{Z} \to \mathrm{Aut}(\mathbf{Z}^3)$. Since S is orientable, $\phi(1) = A \in \mathrm{SL}(3, \mathbf{Z})$, and the action ϕ is defined by

$$\phi(1)(e_i) = \sum_{j=1}^{3} a_{ij} e_j, i = 1, 2, 3,$$

where $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$. We mean by roots of $\phi(1)$ the eigenvalues of $\phi(1)$, the roots of the characteristic polynomial Φ of $\phi(1)$. Since Φ is of the form $\Phi(t) = t^3 - mt^2 + nt - 1$, where m, n are integers, we can see that Φ has no double root except 1 or -1. This is an important Lemma (for which the proof is quite elementary) though it seems to be unknown.

Lemma 2.2. *Let $\Phi(t)$ be a polynomial of the form $\Phi(t) = t^3 - mt^2 + nt - 1 \ (m, n \in \mathbf{Z})$. Then it has a real double root a if and only if $a = 1$ or -1 for which $\Phi(t) = t^3 - 3t^2 + 3t - 1$ or $\Phi(t) = t^3 + t^2 - t - 1$ respectively.*

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Proof. Assume that $\Phi(t)$ has a double root a and another root b. Then we have that $a^2b = 1, 2a + b = m, a^2 + 2ab = n$, from which we deduce that $ma^2 - 2na + 3 = 0, 3a^2 - 2ma + n = 0$; and thus $2(m^2 - 3n)a = mn - 9$. If $m^2 = 3n$, then $m = n = 3$ and $a = 1$. If $m^2 \neq 3n$, then we have that $a = \frac{mn-9}{2(m^2-3n)}$, which is a rational number. Since we have that $2a + \frac{1}{a^2} = m \in \mathbf{Z}$, a must be -1 or 1.

$\square$

Now we classify the group extensions and their corresponding solvmanifolds according to the roots of $\phi(1)$.

1) If $\phi(1)$ has three distinct positive real roots a_1, a_2, a_3, we have linearly independent eigen vectors u_1, u_2, u_3 of a_1, a_2, a_3 respectively. Let $u_i = (u_{i1}, u_{i2}, u_{i3})$, $i = 1, 2, 3$. Then $\{v_1, v_2, v_3\}$, $v_j = (u_{1j}, u_{2j}, u_{3j}), j = 1, 2, 3$ defines an abelian lattice $\mathbf{Z}^3$ of $\mathbf{R}^3$. We define a solvable Lie group $G = \mathbf{R}^3 \rtimes \mathbf{R}$, where the action $\phi : \mathbf{R} \to \mathrm{Aut}(\mathbf{R}^3)$ is defined by

$$\phi(t)((x, y, z)) = (e^{t \log a_1}x, e^{t \log a_2}y, e^{t \log a_3}z),$$

which is a canonical extension of ϕ. Then $\Gamma = \mathbf{Z}^3 \rtimes \mathbf{Z}$ is a lattice of G, and $S = \Gamma \backslash G$ is a solvmanifold. We can see that S is a T^2-bundle over T^2. S has $b_1 = 2$ for the case where one of the roots is 1, and $b_1 = 1$ for the case where none of the roots is 1.

2) If $\phi(1)$ has three distinct real roots two of which are negative, S has a double covering solvmanifold of the above type 1).

3) If $\phi(1)$ has a triple root 1, taking a suitable basis $\{u_1, u_2, u_3\}$ of $\mathbf{R}^3$, $\phi(1)$ is expressed in either of the following forms:

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $G = \mathbf{R}^3 \rtimes \mathbf{R}$, where the action $\bar{\phi} : \mathbf{R} \to \mathrm{Aut}(\mathbf{R}^3)$ is defined by

$$\bar{\phi}(t) = \exp t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix}$$

for the former case, and

$$\bar{\phi}(t) = \exp t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for the latter case. Then, as defined in the case 1), $\{v_0, v_1, v_2, v_3\}$ defines a lattice Γ of G, and $S = \Gamma \backslash G$ is a nilmanifold. We can see that S is a nilmanifold with $b_1 = 2$ for the former case, and a nilmanifold with $b_1 = 3$, a primary Kodaira surface, for the latter case.

4) If $\phi(1)$ has a single root 1, and a double root -1 or non-real complex roots $\beta\,(|\beta| = 1)$, $\bar{\beta}$, then S is a T^2-bundle over T^2 and has $b_1 = 2$. In case $\phi(1)$ has a double root -1, taking a suitable basis $\{u_1, u_2, u_3\}$ of $\mathbf{R}^3$, $\phi(1)$ is expressed in either of the following forms:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Except for the latter case in the above, each corresponding solvmanifold has a complex structure, defining a hyperelliptic surface. For the details of hyperelliptic surfaces we refer to Section 3.3.

We can see that the solvmanifold corresponding to the latter case has a nilmanifold with $b_1 = 3$ as a double covering.

5) If $\phi(1)$ has a positive real root a, and non-real complex roots $\beta\,(|\beta| = b \neq 1)$, $\bar{\beta}$, then S is an Inoue surface of type S. For the details we refer to Section 3.6.

2.3. **The case $k = 1$ and N is non-abelian**

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The group extension has a non-abelian kernel N which is a torsion-free nilpotent group of rank 3. We can see that the extension is split and is determined only by the action $\phi : \mathbf{Z} \to \mathrm{Aut}(N)$. $\phi(1)$ induces an automorphism $\widetilde{\phi}(1)$ of the center $\mathbf{Z}$ of N. $\phi(1)$ also induce an automorphism $\widehat{\phi}(1)$ of $\mathbf{Z}^2 = N/\mathbf{Z}$, that is, $\widehat{\phi}(1) \in \mathrm{GL}(2, \mathbf{Z})$.

1) In the case where $\widetilde{\phi}(1) = \mathrm{Id}$ and $\det \widehat{\phi}(1) = 1$, we have the following:

a) If $\widehat{\phi}(1)$ has positive real roots $a\, (\neq 1), b$, then S is an Inoue surface of type S^+. We refer to Section 3.7 for the details.

b) If $\widehat{\phi}(1)$ has negative real roots $a\, (\neq -1), b$, then S has a double covering solvmanifold of the above type a).

c) If $\widehat{\phi}(1)$ has a double root 1, then we can assume that $\widehat{\phi}(1)$ is of the form
$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$
We can see that $\mathrm{H}^1(\Gamma, \mathbf{Z}) = \mathbf{Z}^2 \oplus \mathbf{Z}/n\mathbf{Z}$. Hence, S is a T^2-bundle over T^2, for which $b_1 = 2$ for the case $n \neq 0$, and $b_1 = 3$ (S is a primary Kodaira surface) for the case $n = 0$.

d) If $\widehat{\phi}(1)$ has a double root -1 or non-real complex roots $\alpha\, (|\alpha| = 1)$, $\bar{\alpha}$, then S has a double covering of the type 1c) and has $b_1 = 1$. In case $\widehat{\phi}(1)$ has a double root -1, we can assume that $\widehat{\phi}(1)$ is of the form
$$\begin{pmatrix} -1 & n \\ 0 & -1 \end{pmatrix}.$$
Except for the case $n \neq 0$ in the above, each corresponding solvmanifold has a complex structure, defining a secondary Kodaira surface. We refer to Section 3.5 for the details of secondary Kodaira surfaces.

2) In the case where $\widetilde{\phi}(1) = -\mathrm{Id}$ and $\det \widehat{\phi}(1) = -1$, it is clear that S has a double covering of a solvmanifold of type 1). We have actually the following:

a) If $\widehat{\phi}(1)$ has real roots $a\,(\neq 1), b$, then S is an Inoue surface of type S^- (see Section 3.7).

b) If $\widehat{\phi}(1)$ has roots 1 and -1, then S is a T^2-bundle over T^2 and has $b_1 = 2$. We refer to Example 3.5 for the details.

Remark 2.2. Γ can be expressed both as a group extension of $\mathbf{Z}^1$ by $\mathbf{Z}^3$ and as that of $\mathbf{Z}^1$ by a non-ableain nilpotent group N if and only if Γ is of the form $N \times \mathbf{Z}^1$. Correspondingly, S belongs to both of the cases (2,2) and (2,3) if and only if S is a nilmanifold with $b_1 = 3$ (a Kodaira surface).

2.4. **Parallelizability of solvmanifolds**

As an application of the classification, we can see the following (cf. [2]).

Proposition 2.3. *Any four-dimensional orientable compact solvmanifold is real parallelizable.*

Proof. In fact, let us consider first a solvmanifold $\Gamma\backslash G$, where $G = \mathbf{R}^2 \rtimes \mathbf{R}$ with the action ϕ defined by

$$\phi(t)(x, y) = (\cos(\pi t)x - \sin(\pi t)y, \sin(\pi t)x + \cos(\pi t)y),$$

and $\Gamma = \mathbf{Z}^2 \rtimes \mathbf{Z}$, which is a lattice of G. Then we have linearly independent left-invariant vector fields X_1, X_2, X_3 on G:

$$X_1 = \cos(\pi t)\tfrac{\partial}{\partial x} + \sin(\pi t)\tfrac{\partial}{\partial y},\ X_2 = -\sin(\pi t)\tfrac{\partial}{\partial x} + \cos(\pi t)\tfrac{\partial}{\partial y}, X_3 = \tfrac{\partial}{\partial t}.$$

As we have seen in the above classification, a four-dimensional solvmanifold S is either of the form $\Gamma\backslash G$, where G is a simply connected solvable Lie group and Γ is a lattice of G, or has a double covering of a solvmanifold $\widehat{S}$ of the above type. For the former case it is clearly parallelizable, and for the latter case since the covering is of the same type as the above, we can construct four linearly independent vector fields on $\widehat{S}$, which are invariant by the covering transformation, in the same way as the above case (though they are not in general left-invariant vector fields on G). □

Remark 2.4. There exists a five-dimensional solvmanifold which is not real parallelizable ([2]).

2.5. **Lattices of solvable Lie groups**.

1) A simply connected solvable Lie group $G = \mathbf{R}^3 \rtimes \mathbf{R}$, where the action $\phi : \mathbf{R} \to \mathrm{Aut}(\mathbf{R}^3)$ is defined by $\phi(t)((x, y, z)) = (e^{at}x, e^{at}y, e^{-2at}z)$, $a(\neq 0) \in \mathbf{R}$, has no lattices, although it has a geometric structure. This is due to the Lemma, and the fact that G has a circle action rotating the first two coordinates. An Inoue surface of type S can be interpreted as a geometry of this type. (for the details we refer to [23]).

2) We have another type of a simply connected, unimodular and non-nilpotent solvable Lie group which has no lattices: Let $\mathfrak{g}$ be a Lie algebra

$$\mathfrak{g} = \{X_1, X_2, X_3, X_4\},$$

where bracket multiplications are defined by

$$[X_4, X_1] = -2X_1, [X_4, X_2] = X_2, [X_4, X_3] = X_3,$$

$$[X_2, X_3] = -X_1, [X_1, X_2] = 0, [X_1, X_3] = 0.$$

Then it can be seen from our classification that the corresponding Lie group G has no lattices.

3. FOUR-DIMENSIONAL SOLVMANIFOLDS – WITH THE CONSTRUCTIONS OF CANONICAL COMPLEX STRUCTURES

We shall study in this section some important classes of solvmanifolds in further detail. In particular we study in separate sections certain classes of solvmanifolds on which we can construct canonical complex structures, determining some explicit classes of complex surfaces.

3.1. Some basic examples

Example 3.1. Let $\Gamma = \mathbf{Z} \rtimes \mathbf{Z}$, where the action $\phi : \mathbf{Z} \to \mathrm{Aut}(\mathbf{Z})$ is defined by $\phi(1) = -1$. Then Γ acts on $\mathbf{R}^2$ as follows:

$$(a, b) \cdot (x, y) = (a + (-1)^b x, b + y),$$

$(x, y) \in \mathbf{R}^2, (a, b) \in \Gamma$. $M = \mathbf{R}^2/\Gamma$ is the Klein Bottle. Let $G = \mathbf{C} \rtimes \mathbf{R}$ be a solvable Lie group defined as follows:

$$(w, t) \cdot (z, s) = (w + e^{\pi i t} z, t + s).$$

Let $D = \{(p + iu, q) \mid p, q \in \mathbf{Z}, u \in \mathbf{R}\}$ and $D' = \{(iu, 0) \mid u \in \mathbf{R}\}$ be closed subgroups of G. Then we have $D = D' \rtimes \Gamma$, and $D \backslash G = (D' \backslash G)/\Gamma = \mathbf{R}^2/\Gamma$.

Example 3.2. Let $G = \mathbf{C} \rtimes \mathbf{R}$ be a solvable Lie group defined as follows:

$$(w, t) \cdot (z, s) = (w + e^{2\pi i t} z, t + s).$$

$\Gamma = \{(p + qi, r) \mid p, q, r \in \mathbf{Z}\}$ is a uniform discrete subgroup of G, which is abelian. $\Gamma \backslash G$ is diffeomorphic to T^3 (a three-dimensional torus).

Example 3.3. Let $\Gamma = \mathbf{Z}^2 \rtimes \mathbf{Z}$, where the action $\phi : \mathbf{Z} \to \mathrm{Aut}(\mathbf{Z}^2)$ is defined by $\phi(1) = A \in \mathrm{SL}(2, \mathbf{Z})$. Assume that A has complex roots $\alpha, \bar{\alpha}$ $(\alpha \neq \bar{\alpha}), |\alpha| = 1$ with the eigen vector $(a, b) \in \mathbf{C}^2$ of α, or a double root -1 with linearly independent eigen vectors $a, b \in \mathbf{C}$ (considering $\mathbf{R}^2$ as $\mathbf{C}$) Since the multiplication by α preserves the lattice spanned by a and b, α must be $e^{i\eta t}, \eta = \frac{2}{3}\pi, \frac{1}{2}\pi$ or $\frac{1}{3}\pi$. Let $G = \mathbf{C} \rtimes \mathbf{R}$ be a solvable Lie group defined as follows:

$$(w, t) \cdot (z, s) = (w + e^{i\eta t} z, t + s),$$

where $\eta = \pi, \frac{2}{3}\pi, \frac{1}{2}\pi$ or $\frac{1}{3}\pi$. Considering Γ as the lattice of G spanned by $(a, 0), (b, 0)$ and $(0, 1)$, $\Gamma \backslash G$ is a solvmanifold, which is a finite quotient of T^3 and has a structure of T^2-bundle over T^1.

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Example 3.4. Let $\Lambda_n = \mathbf{Z}^2 \rtimes \mathbf{Z}$, where the action $\phi : \mathbf{Z} \to \mathrm{Aut}(\mathbf{Z}^2)$ is defined by $\phi(1) = A_n \in \mathrm{GL}(2, \mathbf{Z})$,

$$A_n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

The action ϕ can be extended to $\bar{\phi} : \mathbf{R} \to \mathrm{Aut}(\mathbf{R}^2)$ defined by $\bar{\phi}(t) = A(t) \in \mathrm{GL}(2, \mathbf{R})$,

$$A(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Let $N = \mathbf{R}^2 \rtimes \mathbf{R}$ be a nilpotent Lie group defined by the action $\bar{\phi}$. Then each Λ_n is a lattice of N, and $\Lambda_n \backslash N$ is a nilmanifold.

Example 3.5. Let $\Gamma_n = \Lambda_n \rtimes \mathbf{Z}$, where Λ_n is the same nilpotent group as in Example 3.4, and the action $\phi : \mathbf{Z} \to \mathrm{Aut}(\Lambda_n)$ is defined by $\phi(1) = \tau \in \mathrm{Aut}(\Lambda_n)$,

$$\tau : \begin{pmatrix} 1 & a & \frac{c}{n} \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & a & \frac{-c}{n} \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}$$

Let $G = N \times \mathbf{R}$ be a nilpotent Lie group, where N is the same nilpotent Lie group as in Example 3.4. Γ_n acts as a group of automorphisms on G. $\Gamma_n \backslash G$ is a solvmanifold, which has a nilmanifold with $b_1 = 3$ as a double covering, and has a structure of T^2-bundle over T^2.

3.2. Complex Tori

An n-dimensional torus T^n is a compact homogeneous space of the abelian Lie group $\mathbf{R}^n$: that is, $T^n = \mathbf{Z}^n \backslash \mathbf{R}^n$ where $\mathbf{Z}^n$ is an abelian lattice of $\mathbf{R}^n$ which is spanned by some basis of $\mathbf{R}^n$ as a real vector space. For the case $n = 2m$, the standard complex structure $\mathbf{C}^m$ on $\mathbf{R}^{2m}$ defines a complex structure on T^{2m}. The complex manifold thus obtained is a *complex torus*, which is known to be the only compact complex Lie group. It should be noted that complex structures on T^{2m} differ for different abelian lattices, while the underlying differentiable manifolds are diffeomorphic each other. On the other hand, as was shown in Example 3.2, $T^n (n \geq 3)$ can

admits a structure of non-toral solvmanifold. It is conjectured that complex structures on T^{2m} are only the standard ones (we know that this is valid for $m = 1, 2$ since they are Kählerian).

3.3. Hyperelliptic surfaces

Let $\Gamma = \mathbf{Z}^3 \rtimes \mathbf{Z}^1$, where the action $\phi : \mathbf{Z}^1 \to \mathrm{Aut}(\mathbf{Z}^3)$ is defined by $\phi(1) = A \in \mathrm{SL}(3, \mathbf{Z})$. Assume that A has a single root 1, and a double root -1 with linearly independent eigen vectors of -1 or non-real complex roots β ($|\beta| = 1$), $\bar{\beta}$. We shall see that the class of the corresponding solvmanifolds with canonical complex structures coincides with that of hyperelliptic surfaces.

For a given $A \in \mathrm{SL}(3, \mathbf{Z})$ which satisfies our assumption, we can find a basis $\{u_1, u_2, u_3\}$ of $\mathbf{R}^3$ such that $Au_1 = au_1 - bu_2, Au_2 = bu_1 + au_2, Au_3 = u_3$, where $a = -1$ and $b = 0$ for the case that A has a double root -1, and $a = \mathrm{Re}\,\beta$ and $b = \mathrm{Im}\,\beta$ for the case that A has a non-real complex root β. Let $u_i = (u_{i1}, u_{i2}, u_{i3})$, $i = 1, 2, 3$. Then $\{v_1, v_2, v_3\}$, $v_j = (u_{1j}, u_{2j}, u_{3j}), j = 1, 2, 3$ defines an abelian lattice $\mathbf{Z}^3$ of $\mathbf{R}^3$ which is preserved by a rotation around a fixed axis. In particular, β must be $e^{i\zeta}$ ($\zeta = \frac{2}{3}\pi, \frac{1}{2}\pi$ or $\frac{1}{3}\pi$).

Furthermore, we may assume that $u_{3j} = 0, j = 1, 2$, and A is of the form

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ p & q & 1 \end{pmatrix},$$

where $A' = (a_{ij}) \in \mathrm{SL}(2, \mathbf{Z}), p, q \in \mathbf{Z}$. Since A' has the root -1 (with linearly independent eigen vectors) or β, we can assume that A' is of the form:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

according to the root $e^{i\eta}$ of A', where $\eta = \pi, \frac{2}{3}\pi, \frac{1}{2}\pi$ or $\frac{1}{3}\pi$, respectively.

We now define a solvable Lie group $G = (\mathbf{C} \times \mathbf{R}) \rtimes \mathbf{R}$, where the action $\phi : \mathbf{R} \to \mathrm{Aut}(\mathbf{C} \times \mathbf{R})$ is defined by

$$\phi(t)((z, s)) = (e^{i\eta t} z, s),$$

which is a canonical extension of ϕ. $\Gamma = \mathbf{Z}^3 \rtimes \mathbf{Z}$ clearly defines a lattice of G. Since the action on the second factor $\mathbf{R}$ is trivial, the multiplication of G is defined on $\mathbf{C}^2$ as follows:

$$(w_1, w_2) \cdot (z_1, z_2) = (w_1 + e^{i\eta t} z_1, w_2 + z_2),$$

where $t = \operatorname{Re} w_2$. For each lattice Γ of G, $\Gamma \backslash G$ with canonical complex structure defines a complex surface, actually a hyperelliptic surface.

We can see that there exist seven isomorphism classes of lattices of G, which correspond to seven classes of hyperelliptic surfaces. For each η, take a lattice $\mathbf{Z}^3$ of $\mathbf{R}^3$ spanned by $\{v_1, v_2, v_3\}$ for A with $p = q = 0$. Then we can get a lattice $\mathbf{Z}^3$ spanned by $\{v_1, v_2, v_3'\}$ for A with arbitrary $(p, q) \in \mathbf{Z}^2$, by changing v_3 into $v_3' = sv_1 + tv_2 + v_3$ where $s, t \in \mathbf{Q}$ with $0 \leq s, t < 1$, and $\Gamma = \mathbf{Z}^3 \rtimes \mathbf{Z}$ defines a lattice of the solvable Lie group G. By elementary calculation, we obtain the following seven isomorphism classes of lattices: besides four trivial cases with $(p, q) = (0, 0)$ and $(s, t) = (0, 0)$ for $\eta = \pi, \frac{2}{3}\pi, \frac{1}{2}\pi$ and $\frac{1}{3}\pi$, we have three other cases with $(p, q) = (1, 0)$, and (i) $(s, t) = (\frac{1}{2}, 0)$ for $\eta = \pi$, (ii) $(s, t) = (\frac{1}{3}, \frac{1}{3})$ for $\eta = \frac{2}{3}\pi$, (iii) $(s, t) = (\frac{1}{2}, \frac{1}{2})$ for $\eta = \frac{1}{2}\pi$.

The corresponding solvmanifolds with canonical complex structures coincides with the class of all *hyperelliptic surfaces* (cf. [4]).

3.4. Primary Kodaira surfaces

Let $\Gamma_n = \Lambda_n \times \mathbf{Z}$, and $G = N \times \mathbf{R}$, where Λ_n and N are the nilpotent group and nilpotent Lie group defined in Example 3.4 respectively. $S_n = \Gamma_n \backslash G$ is a nilmanifold with $b_1 = 3$. Expressing the nilpotent Lie group N in Example 3.4 as

$$\begin{pmatrix} 1 & x & s \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

we can define a coordinate change Φ from $N \times \mathbf{R} = \mathbf{R}^3 \times \mathbf{R}$ to $\mathbf{R}^4$:

$$\Phi : ((x, y, s), t) \longrightarrow (x, y, 2s - xy, 2t + \frac{1}{2}(x^2 + y^2)).$$

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Considering $\mathbf{R}^4$ as $\mathbf{C}^2$ the group operation on G in the new coordinate can be expressed as follows:

$$(w_1, w_2) \cdot (z_1, z_2) = (w_1 + z_1, w_2 - i\bar{w}_1 z_1 + z_2).$$

S_n with this complex structure is *a primary Kodaira surface.*

Conversely, any primary Kodaira surface, which is by definition a complex surface with the trivial canonical bundle and $b_1 = 3$, can be written as $\mathbf{C}^2/\Gamma$ where Γ is a properly discontinuous group of affine transformations in the above form ([15]).

3.5. Secondary Kodaira surfaces

Let $\Gamma_n = \Lambda_n \rtimes \mathbf{Z}$, where the action $\phi : \mathbf{Z} \to \mathrm{Aut}(\Lambda_n)$ satisfies the condition that the induced automorphism $\widetilde{\phi}(1)$ of $\mathbf{Z}$ is trivial, that is, $\widetilde{\phi}(1) = \mathrm{Id}$, and the induced automorphism $\widehat{\phi}(1)$ of $\mathbf{Z}^2$ has a double root -1 with linearly independent eigen vectors, or non-real complex roots, $\alpha, \bar{\alpha}\, (\alpha \neq \bar{\alpha}), |\alpha| = 1$.

We shall see that Γ_n can be extended to a solvable Lie group $G = N \rtimes \mathbf{R}$. As we have seen in Section 3.3, α must be $e^{i\eta}$, $\eta = \pi, \frac{2}{3}\pi, \frac{1}{2}\pi$ or $\frac{1}{3}\pi$, and there exists a basis $\{u_1', u_2'\}$ of $\mathbf{R}^2$ such that $Au_1' = au_1' - bu_2', Au_2' = bu_1' + au_2'$, where $A = \widehat{\phi}(1)$, $a = \mathrm{Re}\,\alpha, b = \mathrm{Im}\,\alpha$, and $u_1' = (u_{11}, u_{12}), u_2' = (u_{21}, u_{22})$. The abelian lattice $\mathbf{Z}^2$ of $\mathbf{R}^2$ spanned by $\{v_1', v_2'\}$, where $v_1' = (u_{11}, u_{21}), v_2' = (u_{12}, u_{22})$, is preserved by the automorphism $\psi' : (x, y) \to (ax - by, bx + ay)$ of $\mathbf{R}^2$. We can extend ψ' to an automorphism of N of the form:

$$\psi : (x, y, z) \longrightarrow (ax - by, bx + ay, \frac{z}{n} + h(x, y)),$$

for some polynomial h. As ψ is a group homomorphism of finite order, by simple calculation, we see that $h = \frac{1}{2}b\,(ax^2 - ay^2 - 2bxy)$. We can extend the lattice $\mathbf{Z}^2$ spanned by $\{v_1', v_2'\}$ to a lattice Λ_n spanned by $\{v_1, v_2, v_3\}, v_1 = (u_{11}, u_{21}, u_{31}), v_2 = (u_{12}, u_{22}, u_{32}), v_3 = (0, 0, u_{33})$ for suitable u_{31}, u_{32}, u_{33}, so that Λ_n is preserved by ψ. We now define a solvable Lie group G by

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extending the action $\phi(m) = \psi^m, m \in \mathbf{Z}$ to $\psi(t), t \in \mathbf{R}$, replacing a with $\cos \eta t$ and b with $\sin \eta t$. It is clear that $\Gamma_n = \Lambda_n \rtimes \mathbf{Z}$ defines a lattice of G. If we take the new coordinate as in Section 3.4, the automorphism ψ is expressed as,

$$(z_1, z_2) \longrightarrow (\zeta z_1, z_2)$$

for $\zeta \in \mathbf{C}, |\zeta| = 1$. It follows that the above automorphism is holomorphic with respect to the complex structure defined in Section 3.4. Therefore, $S_n = \Gamma_n \backslash G$ is a finite quotient of a primary Kodaira surface, and S_n with the above complex structure is *a secondary Kodaira surface.*

The classification of secondary Kodaira surfaces are known (cf. [6, 21]): it is by definition a finite quotient of a primary Kodaira surface, which can be also written as $\mathbf{C}^2 / \Gamma$ where Γ is a properly discontinuous group of affine transformations. And all the secondary Kodaira surfaces are constructed in the above way.

3.6. Inoue surfaces of type S

Let $\Gamma = \mathbf{Z}^3 \rtimes \mathbf{Z}$, where the action $\phi : \mathbf{Z} \to \mathrm{Aut}(\mathbf{Z}^3)$ is defined by $\phi(1) = A \in \mathrm{Aut}(\mathbf{Z}^3) = \mathrm{GL}(3, \mathbf{Z})$. Assume that A has complex roots $\alpha, \bar{\alpha}$ and a real root c, where $c \neq 1, |\alpha|^2 c = 1$. Let $(\alpha_1, \alpha_2, \alpha_3) \in \mathbf{C}^3$ be the eigen vector of α and $(c_1, c_2, c_3) \in \mathbf{R}^3$ the eigen vector of c. The set of vectors $\{(\alpha_i, c_i) \in \mathbf{C} \times \mathbf{R} \,|\, i = 1, 2, 3\}$ are linearly independent over $\mathbf{R}$, and defines a lattice $\mathbf{Z}^3$ of $\mathbf{C} \times \mathbf{R}$. Let $G = (\mathbf{C} \times \mathbf{R}) \rtimes \mathbf{R}$ be a solvable Lie group, where the action $\bar{\phi} : \mathbf{R} \to \mathrm{Aut}(\mathbf{C} \times \mathbf{R})$ is defined by $\bar{\phi}(t) : (z, s) \to (\alpha^t z, c^t s)$, which is a canonical extension of ϕ. Then Γ is a lattice of G and $S = \Gamma \backslash G$ is a solvmanifold. Using the diffeomorphism $\mathbf{R} \to \mathbf{R}_+$ defined by $t \to e^{\log ct}$, S can be considered as $\Gamma' \backslash \mathbf{C} \times \mathbf{H}$, where Γ' is a group of automorphisms generated by g_0 and $g_i, i = 1, 2, 3$, which correspond to the canonical generators of Γ. We can see that $g_0 : (z_1, z_2) \to (\alpha z_1, c z_2)$ and $g_i : (z_1, z_2) \to (z_1 + \alpha_i, z_2 + c_i), i = 1, 2, 3$. S with the above complex structure is, by definition, *an Inoue surface of type S.*

3.7. Inoue surfaces of type $\mathbf{S}^{\pm}$

Let $\Gamma_n = \Lambda_n \rtimes \mathbf{Z}$, where the action $\phi : \mathbf{Z} \to \mathrm{Aut}(\Lambda_n)$ satisfies the condition that for the induced action $\widetilde{\phi} : \mathbf{Z} \to \mathrm{Aut}(\mathbf{Z})$, $\widetilde{\phi}(1) = \mathrm{Id}$, and for the induced action $\widehat{\phi} : \mathbf{Z} \to \mathrm{Aut}(\mathbf{Z}^2)$, $\widehat{\phi}(1) = (n_{ij}) \in \mathrm{SL}(2, \mathbf{Z})$ has two positive real roots, a, b $(ab = 1)$. Let $(a_1, a_2), (b_1, b_2) \in \mathbf{R}^2$ be eigen vectors of a, b respectively. Let $G = N \rtimes \mathbf{R}$ be a solvable Lie group, where the action $\bar{\phi} : \mathbf{R} \to \mathrm{Aut}(N)$ is defined by

$$\bar{\phi}(t) : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & a^t x & z \\ 0 & 1 & b^t y \\ 0 & 0 & 1 \end{pmatrix},$$

which is a canonical extension of ϕ. In order to define a lattice Λ_n which is preserved by $\bar{\phi}$, we take $g_1, g_2, g_3 \in N$ as

$$g_1 = \begin{pmatrix} 1 & a_1 & c_1 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & a_2 & c_2 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where c_1, c_2, c_3 are to be determined, satisfying the following conditions:

1) $[g_1, g_2] = g_3^n$

2) $\bar{\phi}(1)(g_1) = g_1^{n_{11}} g_2^{n_{12}} g_3^k$, $\bar{\phi}(1)(g_2) = g_1^{n_{21}} g_2^{n_{22}} g_3^l$, where $k, l \in \mathbf{Z}$.

If we take $g_0 \in N \rtimes \mathbf{R}$ as

$$g_0 = \left(\begin{pmatrix} 1 & 0 & p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, 1 \right), \quad p \in \mathbf{R},$$

then $\{g_0, g_1, g_2, g_3\}$ defines a lattice Γ_n of G, and $S_n = \Gamma_n \backslash G$ is a solvmanifold.

Now, we define a diffeomorphism $\Phi : G = N \rtimes \mathbf{R} \longrightarrow \mathbf{R}^3 \times \mathbf{R}_+$, for an arbitrary $\gamma = p + qi \in \mathbf{C}$ and $\sigma = \log b$, by

$$\Phi : \left(\begin{pmatrix} 1 & y & x \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, t \right) \longrightarrow (x, e^{\sigma t} y + q t, s, e^{\sigma t}).$$

Then considering $\mathbf{R}^3 \times \mathbf{R}_+$ as $\mathbf{C} \times \mathbf{H}$, g_0, g_1, g_2, g_3 are corresponding to the following holomorphic automorphisms of $\mathbf{C} \times \mathbf{H}$,

$$g_0 : (z_1, z_2) \to (z_1 + \gamma, bz_2),$$

$$g_i : (z_1, z_2) \to (z_1 + a_i z_2 + c_i, z_2 + b_i),$$

where $i = 1, 2, 3$ and $a_3 = b_3 = 0$. S_n with the above complex structure is, by definition, *an Inoue surface of type S^+*.

An Inoue surface of type S^- is defined similarly as the case where the action $\phi : \mathbf{Z} \to \mathrm{Aut}(\Lambda_n)$ satisfies the condition that $\widetilde{\phi}(1) = -\mathrm{Id}$, and $\widehat{\phi}(1)$ has a positive and a negative real root. It is clear that an Inoue surface of type S^- has S^+ with $\gamma = 0$ as its double covering surface.

3.8. Complex structures on solvable Lie algebras

We have seen that a four-dimensional solvmanifold S can be expressed, up to finite covering, as $\Gamma \backslash G$ where G is a simply connected solvable Lie group and Γ is a lattice of G. And we have explicitly constructed complex structures on S, as those canonically induced from some left-invariant complex structures on G. In the following list, for each complex surface with diffeomorphism type of solvmanifold, we express the corresponding solvable Lie algebra $\mathfrak{g}$ of G as being generated by $\{X_1, X_2, X_3, X_4\}$ with the specified bracket multiplication. And for each case except 6), the almost complex structure J is defined by $JX_1 = X_2, JX_2 = -X_1, JX_3 = X_4, JX_4 = -X_3$, for which the Nijenhuis tensor $N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$ vanishes.

1) Complex Tori
   $[X_i, X_j] = 0$ $(1 \le i < j \le 4)$.

2) Hyperelliptic Surfaces
   $[X_4, X_1] = -X_2, [X_4, X_2] = X_1$, and all other brackets vanish.

3) Primary Kodaira Surfaces
   $[X_1, X_2] = -X_3$, and all other brackets vanish.

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4) Secondary Kodaira Surfaces

$[X_1, X_2] = -X_3, [X_4, X_1] = -X_2, [X_4, X_2] = X_1$, and all other brackets vanish.

5) Inoue Surfaces of Type S

$[X_4, X_1] = aX_1 - bX_2, [X_4, X_2] = bX_1 + aX_2, [X_4, X_3] = -2aX_3$, and all other brackets vanish, where $a, b \ (\neq 0) \in \mathbf{R}$.

6) Inoue Surfaces of Type S^+ and S^-

$[X_2, X_3] = -X_1, [X_4, X_2] = X_2, [X_4, X_3] = -X_3$, and all other brackets vanish. The almost complex structure J is defined by $JX_1 = X_2, JX_2 = -X_1, JX_3 = X_4 - qX_2, JX_4 = -X_3 - qX_1$, and the Nijenhuis tensor vanishes for this J.

The Inoue Surfaces of Type S^- is not of the form $\Gamma \backslash G$, but has a double covering Inoue surface of type S^+; and its complex structure comes from the one of the above type.

4. PROOF OF THE MAIN THEOREM

In this section we shall prove the main theorem we stated in Section 1; we show actually that a complex surfaces with diffeomorphism type of solvmanifold must be one of the following surfaces: complex tori, hyperelliptic surfaces, Primary Kodaira surfaces, Secondary Kodaira surfaces, and Inoue surfaces. It should be noted that we have already seen in section 3 that all of the above complex surfaces have the structures of four-dimensional solvmanifold. In this section we shall use the standard notations and terminologies in the field of complex surfaces (see [4, 6] for references).

Now let S be a complex surface with diffeomorphism type of solvmanifold. We first remark that since S is parallelizable (see Proposition 2.3) the Euler number c_2 of S vanishes, and the fundamental group of S is abelian if and only if S is a four-dimensional torus ([16]). Let $\kappa(S)$ be the Kodaira

dimension of S. The classification of complex surfaces with $c_2 = 0$ is divided into three cases: $\kappa(S) = -\infty, 0, 1$ (c.f. [4]). In the case where $\kappa(S) = -\infty$, S is a surface of VII_0 or ruled surface of genus 1. The latter surface cannot be diffeomorphic to a solvmanifold since the fundamental group of ruled surface of genus 1 is Z^2 (which is abelian). According to the well-known theorem of Bogomolov (proved by Li, Yau and Zheng [8]), we know that a complex surface of VII_0 with $b_1 = 1$ and $c_2 = 0$ is an Inoue surface or a Hopf surface. Since the fundamental group of the Hopf surface is of the form $H \rtimes \mathbf{Z}$ where H is a finite unitary group (including the trivial case) (cf. [11]), it cannot be the fundamental group of solvmanifold. Hence S must be an Inoue surface. In the case where $\kappa(S) = 0$, S is a complex torus, hyperelliptic surface, or Kodaira surface (primary or secondary Kodaira surface).

In the case where $\kappa(S) = 1$, S is, a (properly) elliptic surface (which is minimal since $c_2 = 0$). Let us first recall some terminologies and fundamental results concerning topology of elliptic surfaces in general. An elliptic surface is a complex surface S together with an elliptic fibration $f : S \to B$ where B is a curve, such that a general fiber $f^{-1}(t), t \in B$ (except finite points $t_1, t_2, ..., t_k$) is an elliptic curve. The base curve B is regarded as a two-dimensional orbifold with multiple points t_i with multiplicity m_i, where $m_i \, (i \geq 2)$ is the multiplicity of the fiber $f^{-1}(t_i)$ $(i = 1, 2, ..., k)$. An elliptic surface S is of the type *hyperbolic, flat (Euclidian), spherical or bad*, according as the orbifold B is of that type. The Euler number $e^{orb}(B)$ of B is by definition $e(B) - \sum_{i=1}^{k} 1 - \frac{1}{m_i}$, where $e(B)$ is the Euler number of B as topological space. We know that B is hyperbolic, flat or spherical according as $e^{orb}(B)$ is negative, 0, or positive. In the case $c_2 = 0$, we can see ([23]) that an elliptic surface S is hyperbolic, flat or spherical according as the Kodaira dimesion $\kappa(S)$ is $1, 0$ or $-\infty$. We now continue our proof for the case $\kappa(S) = 1$. By the above argument, S is a minimal elliptic surface of hyperbolic type. We show that the fundamental group of S is not solvable, and thus S cannot be diffeomorphic to a solvmanifold. We have the

following presentation of $\pi_1(S)$ as a short exact sequence (c.f. [6]):

$$0 \to \mathbf{Z}^2 \to \pi_1(S) \to \pi_1^{orb}(B) \to 0,$$

where $\pi_1^{orb}(B)$ is the fundamental group as two-dimensional orbifold.

Since $\pi_1^{orb}(B)$ is a discrete subgroup of $\mathrm{PSL}(2,\mathbf{R})$, it contains a torsion-free subgroup Γ of finite index, such that Γ is the fundamental group of a finite orbifold covering $\widetilde{B}$ of B, which is a close surface of genus $g \geq 2$ (or a Riemann surface of hyperbolic type) (cf. [18]). We know that Γ is represented as a goup with generator $\{a_1, a_2, ..., a_g, b_1, b_2, ..., b_g\}$ and relation $\prod_{i=1}^{g}[a_i, b_i] = 1$, which is not solvable for $g \geq 2$ (cf. [5]). It follows that $\pi_1(S)$ cannot be solvable since the quotient groups and subgroups of a solvable group must be solvable. This completes the proof of Main Theorem. $\quad\square$

We state the above result as the theorem, with its immediate consequence as a proposition.

Theorem 4.1 (Main Theorem). *The complex surfaces with diffeomorphism type of solvmanifolds are all of the complex tori, hyperelliptic surfaces, Primary Kodaira surfaces, Secondary Kodaira surfaces and Inoue surfaces.*

Combined with the results of section 3, we obtain the following result.

Proposition 4.2. *Every complex structure J on a four-dimensional solvmanifold S is canonical: that is, up to finite covering, S can be written as $\Gamma\backslash G$, where G is a simply connected solvable Lie group, Γ is a lattice of G, and J is the complex structure canonically induced from a left-invariant complex structure on G.*

Conjecture. It is natural to conjecture that the above proposition holds for solvmanifolds of general dimensional. It should be noted that this conjecture is closely related to the general conjecture on Kaehlerian solvmanifolds (see [12] for the detail).

5. APPENDIX – COMPLEX STRUCTURES ON FOUR-DIMENSIONAL COMPACT HOMOGENEOUS SPACES

In this section we determine complex surfaces with diffeomorphism type of compact homogeneous spaces. It is known (due to V. V. Gorbatsevich [9], also see [10]) that a four-dimensional compact homogeneous space is diffeomorphic to one of the following types: (1) $\prod S^{k_i}$ (up to finite quotient), where $k_i \geq 1$ with $\sum k_i = 4$, (2) $\mathbf{C}\mathrm{P}^2$, (3) Solvmanifold, (4) $S^1 \times \Gamma \backslash \widetilde{\mathrm{SL}}_2(\mathbf{R})$, where $\widetilde{\mathrm{SL}}_2(\mathbf{R})$ is the universal covering of $\mathrm{SL}_2(\mathbf{R})$ and Γ is a lattice of $\widetilde{\mathrm{SL}}_2(\mathbf{R})$. We can determine, from the above result, complex structures on four-dimensional compact homogeneous spaces in the following list:

S	b_1	Complex Structure	κ
$S^2 \times T^2$	2	Ruled Surface of genus 1	
$S^1 \times_{\mathbf{Z}_m} S^3/H$	1	Hopf Surface	
$S^2 \times S^2$	0	Hirzebruch Surface of even type	$-\infty$
$\mathbf{C}\mathrm{P}^2$	0	Complex Projective Space	
	1	Inoue Surface	
Solvmanifold	4	Complex Torus	
	3	Primary Kodaira Surface	
	2	Hyperelliptic Surface	0
	1	Secondary Kodaira Surface	
$S^1 \times \Gamma \backslash \widetilde{\mathrm{SL}}_2(\mathbf{R})$	odd	Properly Elliptic Surface	1

where κ is the Kodaira dimension of S, H is a finite subgroup of SU(2) acting freely on S^3.

It is well-known (due to A. Borel and J. P. Serre) that S^4 has no almost complex structure. For the case of $S^2 \times T^2$, it is known (due to T. Suwa [20]) that a complex surface is diffeomorphic to a S^2-bundle over T^2 if and only if it is a ruled surface of genus 1. We can see this also from the recent result (due to R. Friedman and Z. B. Qin [7]) that the Kodaira dimension of complex algebraic surface is invariant up to diffeomorphism. To be more precise, there exist two diffeomorphism types of ruled surfaces of genus 1: the

trivial one and the non-trivial one (which correspond to two diffeomorphism types of S^2-bundles over T^2), and the latter is not of homogeneous space form (see [20]). For the case of $S^1 \times S^3$, K. Kodaira showed [14] that a complex surface diffeomorphic to a finite quotient of $S^1 \times S^3$ is a Hopf surface. Generally a Hopf surface is diffeomorphic to a fiber bundle over S^1 with fiber S^3/U, defined by the action $\rho : \pi_1(S^1) \to N_{\mathrm{U}(2)}(\mathrm{U})$ with $\rho(1)$ being cyclic of order m, where U is a finite subgroup of U(2) acting freely on S^3: that is, $S = S^1 \times_{\mathbf{Z}_m} S^3/U$ (cf. [11]). We can see that a Hopf surface is of homogeneous space form if and only if U is a finite subgroup of SU(2). Let $G = \mathrm{SU}(2) \times S^1$, which is a compact Lie group structure on $S^3 \times S^1$. Take a finite subgroup $\Delta = H \rtimes \mathbf{Z}_m$ of G, where H is a finite subgroup of SU(2), $\mathbf{Z}_m$ is a finite cyclic subgroup of G generated by c:

$$c = (\tau, \xi), \tau = \begin{pmatrix} \xi^{-1} & 0 \\ 0 & \xi \end{pmatrix}, \xi^m = 1,$$

and τ belongs to $N_{\mathrm{SU}(2)}(H)$. S is a fiber bundle over S^1 with fiber S^3/H, which has a canonical complex structure, defining a Hopf surface. It should be noted that if τ does not belong to H and $m \geq 2$, then S is a non-trivial bundle. Conversely, given a Hopf surface S with fiber S^3/H, defined by the action ρ, we can assume ([11]) that $\rho(1)$ is a diagonal matrix, all of which entries are m-th roots of 1. Then we can see that $\rho(1)$, which belongs to $N_{\mathrm{U}(2)}(H)$, actually belong to $N_{\mathrm{SU}(2)}(H)$. Hence S is diffeomorphic to the one constructed above. For the case of $S^2 \times S^2$, it was shown (due to Z. B. Qin [17]) that a complex surface diffeomorphic to $S^2 \times S^2$ must be a Hirzebruch surface of even type, which is by definition a ruled surface of genus 0 with diffeomorphism type $S^2 \times S^2$. As is well-known, there exist two diffeomorphism types of ruled surfaces of genus 0: $S^2 \times S^2$ and $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$ (which correspond to two diffeomorphism types of S^2-bundles over S^2). A Hirzebruch surface of odd type is the surface of the latter type. We can see that no non-trivial finite quotient of $S^2 \times S^2$ has complex structure. It is well-known (due to S. -T. Yau) that $\mathbf{CP}^2$ can have only the standard

complex structure. We have studied in detail the case of solvmanifolds in this paper. The complex surfaces with diffeomorphism type of solvmanifolds are Inoue surfaces for $\kappa = -\infty$ and all of those with $c_2 = 0$ for $\kappa = 0$. For the case of $S^1 \times \Gamma\backslash\widetilde{\text{SL}}_2(\mathbf{R})$, C. T. C. Wall showed [23] that it admits a canonical complex structure, which define a properly elliptic surface with $b_1 = $ odd and $c_2 = 0$; and conversely any such surface with no singular fibers is diffeomorphic to $S^1 \times \Gamma\backslash\widetilde{\text{SL}}_2(\mathbf{R})$ for some lattice Γ.

Acknowledgments

REFERENCES

[1] L. Auslander, *An exposition of the structure of solvmanifolds I II*, Bull. Amer. Math. Soc., **79** (1973), No. 2, 227-261, 262-285.

[2] L. Auslander and R. H. Szczarba, *Characteristic Classes of Compact Solvmanifolds*, Ann. of Math, **76** (1962), 1-8.

[3] L. Auslander and H. Szczarba, *Vector bundles over noncompact solvmanifolds*, Amer. J. of Math., **97** (1975), 260-281.

[4] W. Barth, C. Peters and Van de Ven, *Compact Complex Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 4, Springer-Verlag, 1984.

[5] K. T. Chen, *Algeras of iterated path integrals and fundamental groups*, Trans. Amer. Math. Soc., bf 156(1971), 359-379.

[6] R. Friedman and J. W. Morgan, *Smooth Four-Manifolds and Complex Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 27, Springer-Verlag, 1994.

[7] R. Friedman and Z. B. Qin, *On complex surfaces diffeomorphic to rational surfaces*, Invent. Math., **120** (1995), 81-117.

[8] J. Li, S. T. Yau and F. Zheng, *A simple proof of Bogomolov's theorem on class VII_0 surfaces with $b_2 = 0$*, Illinois J. Math., **34** (1990), 217-220.

[9] V. V. Gorbatsevich, *On the classification of four-dimensional compact homogeneous spaces*, Usp. Mat. Nauk., **32** (1977), No. 2, 207-208 (Russian).

[10] V. V. Gorvatsevich and A. L. Onishchik, *Lie Transformation Groups*, Encyclopaedia of Mathematical Sciences, Vol. 20, Lie Groups and Lie Algebras I, Springer-Verlag, 1994.

[11] K. Hasegawa, *Deformations and diffeomorphism types of generalized Hopf manifolds*, Illinois J. Math., **37** (1993), 643-651.

[12] K. Hasegawa, *A class of compact Kählerian solvmanifolds and a general conjecture*, Geometriae Dedicata, **78** (1999), 253-258.

[13] M. Inoue, *On surfaces of class VII_0*, Inventiones math., **24** (1974), 269-310.

[14] K. Kodaira, *Complex structures on $S^3 \times S^1$*, Proc. Nat. Acad. Sci., **55** (1966), 240-243.

[15] K. Kodaira, *On the structure of compact complex analytic surfaces, I.*, Amer. J. Math., **86** (1964), 751-798; *II.*, Am. J. Math., **88** (1966), 682-721.

[16] G. D. Mostow, *Factor spaces of solvable Lie groups*, Ann. of Math., **60** (1954), 1-27.

[17] Z. B. Qin, *Complex structure on certain differential 4-manifolds*, Topology, **32** (1993), No. 3, 551-566.

[18] P. Scott, *The Geometries of 3-manifolds*, Bull. London Math. Soc., **15** (1983), 401-487.

[19] K. Sakamoto and S. Fukuhara, *Classification of T^2 bundles over T^2*, Tokyo J. Math., bf 6 (1983), 311-327.

[20] T. Suwa, *Ruled surfaces of genus* 1, J. Math. Soc. Japan, **21** (1969), 291-311.

[21] T. Suwa, *Compact quotient spaces of* $\mathbf{C}^2$ *by affine transformation groups*, J. Differential Geometry, **10** (1975), 239-252.

[22] M. Ue, *Geometric 4-manifolds in the sense of Thurston and Seifert 4-manifolds I*, J. Math. Soc. Japan, **42** (1990), 511-540.

[23] C. T. C. Wall, *Geometric structures on compact complex analytic surfaces*, Topology, **25** (1986), No. 2, 119-153.

[24] C. T. C. Wall, *Geometries and geometric structures in real dimension 4 and complex dimension* 2, Lecture Notes in Math., **1167** (1986), 118-153.

[25] H. C. Wang, *Discrete subgroups of solvable Lie groups I*, Ann. of Math., **64** (1956), 1-19.

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